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A dielectric flow solution with maximal supersymmetry

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Abstract: We obtain a solution to eleven-dimensional supergravity that consists of M2-branes embedded in a dielectric distribution of M5-branes. Contrary to normal expectations, this solution has maximal supersymmetry for a brane solution (i.e. sixteen supercharges). While the solution is constructed using gauged supergravity in four dimensions, the complete eleven-dimensional solution is given. In particular, we obtain the Killing spinors explicitly, and we find that they are characterised by a duality rotation of the standard Dirichlet projection matrix for M2-branes.

Keywords: M-Theory, Superstring Vacua, AdS-CFT and dS-CFT Correspondence.

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1. Introduction

The classification of supersymmetric backgrounds with tensor gauge field fluxes remains a poorly understood subject, which we believe will ultimately yield some beautiful and rich structures. These sorts of backgrounds are essential to the study of holographic RG flows, and yet we still do not know enough about them to construct some of the most basic of holographic flows. There are many examples, and some simple, unifying ideas, like the "harmonic rule," and more recently, the application of G-structures, but the former has rather limited application, and although the latter idea has led to some very interesting insights, it is still under development (see, for example, [1]–[3]).

There are some surprising gaps in our knowledge. For example, there is the construction of the most general $\mathcal{N} = 2^*$ holographic flow from the $\mathcal{N} = 4$ fixed point theory. The wilsonian effective action of the field theory is known [4], but the corresponding holographic flow is only known for one point on the (infinite dimensional) Coulomb branch [5]. There have been some recent examples that go beyond the usual harmonic rule, and go against the conventional wisdom in that they are combinations of branes and anti-branes, and yet preserve some supersymmety (see, for example, [6, 7]). There have also been some recent conjectures about continuous duality symmetries that trade pure metric deformations for RR-fluxes, while preserving supersymmetry [8].

Our purpose here is to provide, and study in detail, an example that touches upon all the foregoing issues. Our example is unusual in that it is a flow solution that has *six-*

teen supersymmetries, and is not merely some set of parallel "harmonic" branes. Indeed, it looks more like M2-branes embedded in a dielectric distribution M5-branes. Unlike previous examples, the solution presented here preserves half the supersymmetries, and indeed it has the maximal supersymmetry possible for a brane solution. Thus our solution provides an interesting extension of the conventional wisdom, and this will need to be properly incorporated into any classification. Indeed, since we will exhibit all the supersymmetries explicitly in eleven dimensions, it would be interesting to see if the ideas of G-structures could be used to understand this example more deeply, and perhaps to generalise it.

Our solution is a little reminiscent of the solutions of [7], in which supersymmetric combinations of $D2$ branes and anti-$D2$ branes are generated by introducing electric and magnetic fields on the world volume. The result is a form of "dielectric effect" in which $D0$ branes and fundamental strings are dissolved into the $D2$-brane, making a composite 1/4 supersymmetric system. Here we present a continuous family of solutions that starts with the standard, near-brane, large N limit of M2-branes, and then turns on a 3-form potential that appears to be sourced by a dielectric family of M5-branes that contain the original M2-branes. Turning on the additional 3-form potential preserves the amount of supersymmmetry, but the actual supersymmetry parameters rotate as the parameter is changed.

Another motivation in studying this solution was a conjecture arising out of the holographic description of flows from $\mathcal{N} = 2$ quiver gauge theories. In [8] it was argued that a particular subsector of such gauge theories was holographically dual to a particular family of gauged $\mathcal{N} = 4$ supergravity theories in five dimensions. One consequence of this work was the prediction of a global $\mathrm{SU}(p + 1, 1)$ symmetry that acts on the large N, A_p quiver gauge theories. In particular, the $\mathrm{SU}(p)$ subgroup of this symmetry acts on the non-trivial $\mathcal{N} = 1$ superconformal fixed points, making them into a $\mathbb{CP}^p$ fixed surface. From the field theory point of view such a symmetry is plausible (at large N), and from the five-dimensional supergravity perspective, this symmetry is obvious [8]. On the other hand, from the perspective of the underlying, ten-dimensional IIB theory, such a symmetry involves trading Kähler moduli of blown-up $\mathbb{P}^1$'s for topologically trivial fluxes, while preserving supersymmetry. Unfortunately, the ten-dimensional solution seems to be too complicated for explicit construction. On the other hand, the general issue of "rotating metric deformations into fluxes" is important, and we would like to understand it more deeply; hence this example.

As with a number of interesting, and non-trivial examples, the origin of this one is gauged supergravity. In section 2 we use gauged supergravity in four dimensions to obtain a one-parameter family of flows that preserve sixteen supersymmetries. These flows preserve an $\mathrm{SO}(4) \times \mathrm{SO}(4)$ symmetry, and involve a four-dimensional scalar and pseudo-scalar. The parameter, ζ, remains fixed along the flow, and changing ζ rotates the scalar into the pseudo-scalar. From the eleven-dimensional perspective of M-theory, a four-dimensional scalar comes from the internal metric, while the pseudo-scalar comes from the internal 3-form potential, $A^{(3)}_{mnp}$. Thus a "trivial" rotation of the four-dimensional configuration has a highly non-trivial effect in eleven dimensions.

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Fortunately, the uplift formulae are known [9]–[11], and they have a particularly simple form [11] for the fields we consider here.[1] In section 3, we use these results to obtain the corresponding eleven-dimensional backgrounds. In section 4 we compute explicitly the sixteen supersymmetries in eleven dimensions. In section 5 we show how the $\zeta = 0$ flow has a standard "harmonic" form, and represents a distribution of M2-branes that have been spread out, with constant density, in a 4-ball in the $\mathbb{R}^4 \subset \mathbb{R}^8$ transverse to the branes. The $\zeta = \pi$ flow is similarly simple: the branes a spread uniformly in a 4-ball in the orthogonal $\mathbb{R}^4 \subset \mathbb{R}^8$. In the latter part of section 5, we examine the solution for the intermediate values of ζ. The internal gauge fields are reminiscent of the harmonic rule for M5-branes, and they spread into $\mathbb{R}^4 \times \mathbb{R}^4 \subset \mathbb{R}^8$, with one distribution of M5-branes in the first $\mathbb{R}^4$ factor, and another distribution of M5-branes in the second $\mathbb{R}^4$. We also show how changing ζ rotates the supersymmetry parameters.

2. The flow solutions in four-dimensional supergravity

Following [12, 13], we define the action of the $E_{7(7)}$ by

$$\delta z_{IJ} = \Sigma_{IJKL}\, z^{KL} \delta z^{IJ} = \Sigma^{IJKL}\, z_{KL}\,, \tag{2.1}$$

where indices are raised and lowered by complex conjugation, and where one has

$$\Sigma_{IJKL} = \overline{(\Sigma^{IJKL})} = \frac{1}{24}\, \varepsilon_{IJKLPQRS}\, \Sigma^{PQRS}\,.$$

In this formulation, we are going to consider a very simple $\mathrm{SL}(2, \mathbb{R}) \subset E_{7(7)}$ defined by

$$\Sigma_{IJKL} = 24 \left(z_0 \delta^{1234}_{[IJKL]} + \bar{z}_0\, \delta^{5678}_{[IJKL]} \right)\,, \tag{2.2}$$

where z_0 is a complex parameter corresponding to the non-compact generators of the $\mathrm{SL}(2, \mathbb{R})$.

This subset of the scalars has a manifest $\mathrm{SO}(4) \times \mathrm{SO}(4)$ invariance acting on the indices $1, \ldots, 4$ and $5, \ldots, 8$ separately. This invariance is a subgroup of the full $\mathrm{SO}(8)$ gauge symmetry, and so it will be an invariance of the solutions that we obtain here.

It is important to note that the $\mathrm{U}(1)$ subgroup of the $\mathrm{SL}(2, \mathbb{R})$ is *not* a subgroup of the $\mathrm{SO}(8)$ gauge symmetry: this $\mathrm{U}(1)$ rotates scalars into pseudo-scalars in the four-dimensional theory, and it is the $\mathrm{U}(1)$ in the $\mathrm{SU}(8) \subset E_{7(7)}$ defined by

$$U = \mathrm{diag}(+i\zeta, +i\zeta, +i\zeta, +i\zeta, -i\zeta, -i\zeta, -i\zeta, -i\zeta)\,. \tag{2.3}$$

This $\mathrm{U}(1)$ is thus not an *a priori* symmetry of the solution. However, it is a duality symmetry of a truncated form of the $\mathcal{N} = 8$ theory.

[1] Strictly speaking, the uplift formulae for $\mathcal{N} = 8$ gauged supergravity obtained in [9, 10] were not explicit in the 4-form sector, while the uplift formulae for $\mathcal{N} = 4$ gauged supergravity obtained in [11] were complete and explicit in the entire bosonic sector, but did not include the fermions. Thus our present work provides a non-trivial check on the uplift formulae, since we explicitly verify that the flow solution in the four-dimensional gauged $\mathcal{N} = 4$ supergravity does indeed exhibit the expected supersymmetry after uplifting to eleven dimensions.

To be more explicit, if we truncate the entire $\mathcal{N}=8$ theory to the singlets of the second of the SO(4) factors in the SO(4) $\times$ SO(4) symmetry, then this will truncate the $\mathcal{N}=8$ supergravity down to $\mathcal{N}=4$ gauged SO(4) supergravity (in which the gauged SO(4) is the first of the SO(4) factors). In this truncated theory, the U(1) acts as an "electric-magnetic" duality symmetry on the gauge fields of the supergravity, and is thus a symmetry of the equations of motion.

One can use this U(1) action to parametrise z_0:

$$z_0 = \frac{1}{2}\,\alpha\, e^{i\,\zeta}\,. \tag{2.4}$$

and then the scalar kinetic term takes the form

$$(\partial_\mu\, \alpha)^2 + \frac{1}{4}\, \sinh^2 2\alpha\, (\partial_\mu\, \zeta)^2\,. \tag{2.5}$$

On this sector the scalar potential is extremely simple, namely

$$\mathcal{P} = -\frac{1}{L^2}\,(2+\cosh 2\alpha)\,, \tag{2.6}$$

where we have replaced the usual supergravity gauge coupling, g, according to

$$g = \frac{1}{\sqrt{2}\,L}\,. \tag{2.7}$$

To find the supersymmetries one computes the SU(8) tensor, A_1^{ij}, that appears in the gravitino transformation rule. One finds that it is real and proportional to the identity matrix:

$$A_1^{ij} = \cosh\alpha\, \delta^{ij}\,. \tag{2.8}$$

This means that if we have any supersymmetry then we have maximal supersymmetry. It also implies that there is a superpotential,

$$\mathcal{W} = \cosh\alpha\,, \tag{2.9}$$

and indeed one has

$$\mathcal{P} = \frac{1}{L^2}\left|\frac{\partial \mathcal{W}}{\partial\alpha}\right|^2 - \frac{3}{L^2}\,|\mathcal{W}|^2\,. \tag{2.10}$$

The consequence of all this is that if we use the usual flow metric,

$$ds^2_{1,3} = e^{2A(r)}\eta_{\mu\nu}\,dx^\mu dx^\nu + dr^2\,, \tag{2.11}$$

where $\eta_{\mu\nu}$ is the flat, Poincaré invariant metric of the M2-brane, then the following equations of motion yield a *maximally* supersymmetric flow:

$$\frac{d\alpha}{dr} = -\frac{1}{L}\,\frac{\partial\mathcal{W}}{\partial\alpha}\,, \qquad \frac{d\zeta}{dr} = 0\,, \qquad \frac{dA}{dr} = \frac{1}{L}\,\mathcal{W}\,. \tag{2.12}$$

These have the solution

$$e^\alpha = \coth\frac{r}{2L}\,, \qquad e^A = \sinh\frac{r}{L} = \frac{1}{\sinh\alpha}\,, \qquad \zeta = \text{const.} \tag{2.13}$$

The remarkable fact is that the potential, the superpotential and the flows are all independent of the choice of ζ, and in particular the flow is maximally supersymmetric for all choices of ζ. The U(1) rotation by ζ is only a duality symmetry of $\mathcal{N}=4$ supergravity, and thus might, *a priori*, be expected to preserve the four supersymmetries of that theory. However, we have shown a stronger result: this duality symmetry preserves all the supersymmetries of the $\mathcal{N}=8$ theory. Again, we stress that rotations in ζ are generically *not* symmetries of the gauged supergravity in four dimensions: this rotation takes scalars into pseudo-scalars. In the linearised eleven-dimensional theory, such a rotation takes metric modes into tensor gauge field modes. This symmetry is thus rather unexpected, and must have interesting geometric consequences.

3. Consistent truncations/$\mathcal{N}=4$ supergravity

Since our flow lies entirely within the $\mathcal{N}=4$, SO(4) gauged supergravity theory, we can use the remarkably simple "uplift" formulae of [11] to find the corresponding solution in eleven dimensions. We therefore review the results for this particular Kaluza-Klein reduction Ansatz for obtaining $\mathcal{N}=4$ SO(4) gauged supergravity in four dimensions from an S^7 reduction of eleven-dimensional supergravity. For our present purposes we can set the Yang-Mills fields to zero, since these do not participate in the four-dimensional supergravity solution that we are considering here. After doing this, the results for the consistent embedding obtained in [11] are as follows. The reduction Ansatz for the eleven-dimensional metric is given by

$$d\hat{s}_{11}^2 = \Omega^2\, ds_4^2 + 2g^{-2}\,\Omega^2\, d\theta^2 + \frac{1}{2}g^{-2}\,\Omega^2 \left[\frac{c^2}{Y}\sum_i (\sigma^i)^2 + \frac{s^2}{\widetilde{Y}}\sum_i (\tilde{\sigma}_i)^2\right], \tag{3.1}$$

where

$$\begin{aligned} &\widetilde{X} \equiv X^{-1}\,q\,, \qquad q^2 \equiv 1+\chi^2\,X^4\,, \qquad c\equiv\cos\theta\,, \qquad s\equiv\sin\theta \\ &\Omega \equiv \left[Y\,\widetilde{Y}\right]^{\frac{1}{6}}\,, \qquad Y\equiv (c^2\,X^2+s^2)\,, \qquad \widetilde{Y}\equiv (s^2\,\widetilde{X}^2+c^2)\,. \end{aligned} \tag{3.2}$$

The constant g is the supergravity gauge coupling constant. The three quantities σ_i are left-invariant 1-forms on $S^3=\mathrm{SU}(2)$, and the three $\tilde{\sigma}_i$ are left-invariant 1-forms on a second S^3. They satisfy[2]

$$d\sigma_i = \frac{1}{2}\epsilon_{ijk}\,\sigma_j\wedge\sigma_k\,, \qquad d\tilde{\sigma}_i = \frac{1}{2}\epsilon_{ijk}\,\tilde{\sigma}_j\wedge\tilde{\sigma}_k\,. \tag{3.3}$$

Explicitly, in terms of Euler angles, φ_i, one has

$$\begin{aligned} \sigma_1 &\equiv \cos\varphi_3\,d\varphi_1 + \sin\varphi_3\,\sin\varphi_1\,d\varphi_2\,, \\ \sigma_2 &\equiv \sin\varphi_3\,d\varphi_1 - \cos\varphi_3\,\sin\varphi_1\,d\varphi_2\,, \\ \sigma_3 &\equiv \cos\varphi_1\,d\varphi_2 + d\varphi_3\,, \end{aligned} \tag{3.4}$$

with a similar expression for the $\tilde{\sigma}_i$ in terms of a second set of Euler angles, $\tilde{\varphi}_j$.

[2] Note that we have changed orientation conventions relative to those used in [11].

The remaining bosonic fields of the $\mathcal{N}=4$ supermultiplet are the dilaton ϕ and the axion χ. The dilaton parameterises the quantity X appearing in (3.1) and (3.2), being related to it by

$$X = e^{\frac{1}{2}\phi}\,.$$

With the Yang-Mills fields set to zero, the reduction Ansatz for $\hat{F}_{(4)}$ is given by

$$\hat{F}_{(4)} = -g\,\sqrt{2}\,U\,\epsilon_{(4)} - \frac{4s\,c}{g\,\sqrt{2}}\,X^{-1}\,{*dX}\wedge d\theta + \frac{\sqrt{2}s\,c}{g}\,\chi\,X^4\,{*d\chi}\wedge d\theta + \hat{F}'_{(4)}\,, \tag{3.5}$$

where

$$U = 1 + Y + \widetilde{Y} = X^2\,c^2 + \widetilde{X}^2\,s^2 + 2\,,$$

and $\hat{F}'_{(4)} = d\hat{A}'_{(3)}$, with

$$\hat{A}'_{(3)} = f\,\epsilon_{(3)} + \tilde{f}\,\tilde{\epsilon}_3\,. \tag{3.6}$$

Here $\epsilon_{(3)} = \frac{1}{6}\epsilon_{ijk}\,\sigma_i\wedge\sigma_j\wedge\sigma_k$ and $\tilde{\epsilon}_{(3)} = \frac{1}{6}\epsilon_{ijk}\,\tilde{\sigma}_i\wedge\tilde{\sigma}_j\wedge\tilde{\sigma}_k$. The functions f and $\tilde{f}$ are given by

$$\begin{aligned} f &= -\frac{1}{2\sqrt{2}}\,g^{-3}\,c^4\chi\,X^2\,(c^2\,X^2 + s^2)^{-1}\,,\\ \tilde{f} &= \frac{1}{2\sqrt{2}}\,g^{-3}\,s^4\,\chi\,X^2\,(s^2\,\widetilde{X}^2 + c^2)^{-1}\,. \end{aligned} \tag{3.7}$$

The field strength contribution $\hat{F}'_{(4)}$ is therefore given by

$$\begin{aligned} \hat{F}'_{(4)} = {}& \frac{\partial f}{\partial\chi}\,d\chi\wedge\epsilon_{(3)} + \frac{\partial f}{\partial X}\,dX\wedge\epsilon_{(3)} + \frac{\partial f}{\partial\theta}\,d\theta\wedge\epsilon_{(3)} + \\ & + \frac{\partial\tilde{f}}{\partial\chi}\,d\chi\wedge\tilde{\epsilon}_{(3)} + \frac{\partial\tilde{f}}{\partial X}\,dX\wedge\tilde{\epsilon}_{(3)} + \frac{\partial\tilde{f}}{\partial\theta}\,d\theta\wedge\tilde{\epsilon}_{(3)}\,. \end{aligned} \tag{3.8}$$

To make contact with gauged supergravity we need to reparametrise the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset space in terms of its Lie algebra generators. The group manifold of $\mathrm{SL}(2,\mathbb{R})$ can be thought of as the hyperboloid

$$X_0^2 + X_1^2 - X_2^2 = 1$$

in $\mathbb{R}^3$. In the reduction Ansatz above, we have used horospherical coordinates in which one has

$$X_0 + X_1 = e^{\phi}\,,\qquad X_0 - X_1 = e^{-\phi} + e^{\phi}\,\chi^2\,,\qquad X_2 = e^{\phi}\,\chi\,.$$

If one goes to a complex ($\mathrm{SU}(1,1)$) basis, and writes the non-compact Lie-algebra generator of $\mathrm{SL}(2,\mathbb{R})$ as in (2.4), then one has $X_0 = \cosh 2\alpha$, $X_1 = \sinh 2\alpha\,\cos\zeta$ and $X_2 = \sinh 2\alpha\,\sin\zeta$, from which one obtains

$$e^{\phi} = \cosh 2\alpha + \sinh 2\alpha\cos\zeta\,,\qquad \chi = \frac{\sinh 2\alpha\sin\zeta}{\cosh 2\alpha + \sinh 2\alpha\cos\zeta}\,. \tag{3.9}$$

With these changes of variable, and using (2.7), the metric may be written in terms of the frames

$$\begin{aligned} e^1 &= \Omega e^A dt\,,\qquad e^2 = \Omega e^A dx\,,\qquad e^3 = \Omega e^A dy\,,\\ e^4 &= \Omega dr\,,\qquad e^5 = 2L\Omega d\theta\,,\\ e^{j+5} &= L\Omega Y^{-\frac{1}{2}}\cos\theta\sigma_j\,,\qquad e^{j+8} = L\Omega\widetilde{Y}^{-\frac{1}{2}}\sin\theta\tilde{\sigma}_j\,,\qquad j=1,2,3\,, \end{aligned} \tag{3.10}$$

where the functions Y and $\widetilde{Y}$ are given by

$$\begin{aligned} Y &= \sin^2\theta + \cos^2\theta(\cosh 2\alpha + \cos\zeta \sinh 2\alpha)\,, \\ \widetilde{Y} &= \cos^2\theta + \sin^2\theta(\cosh 2\alpha - \cos\zeta \sinh 2\alpha)\,. \end{aligned} \tag{3.11}$$

The metric defined by (3.10) has a manifest symmetry of SO(4) × SO(4), and there is also an interchange symmetry

$$\theta \to \frac{\pi}{2} - \theta\,, \qquad \alpha \to -\alpha\,. \tag{3.12}$$

The 3-form potential is given by

$$\begin{aligned} A^{(3)} &= \frac{k^3\, Z}{\sinh^3\alpha} dt \wedge dx \wedge dy + L^3 \sin\zeta \sinh 2\alpha \, \times \\ &\times \left(\frac{\cos^4\theta}{Y} \sigma_1 \wedge \sigma_2 \wedge \sigma_3 - \frac{\sin^4\theta}{\widetilde{Y}} \tilde{\sigma}_1 \wedge \tilde{\sigma}_2 \wedge \tilde{\sigma}_3 \right), \end{aligned} \tag{3.13}$$

where the function Z is defined by

$$Z = \frac{1}{2\cosh\alpha}(Y + \widetilde{Y}) = \cosh\alpha + \cos\zeta \sinh\alpha \cos 2\theta\,. \tag{3.14}$$

This configuration satisfies equations of motion

$$R_{MN} + R\, g_{MN} = \frac{1}{12} F^{(4)}_{MPQR} F_N^{(4)PQR}\,, \qquad d * F^{(4)} = \frac{1}{2} F^{(4)} \wedge F^{(4)}\,, \tag{3.15}$$

where $*$ is defined using $\epsilon^{1\cdots 11} = 1$.

4. Supersymmetry

The gravitino variation is

$$\delta\psi_\mu = \nabla_\mu \varepsilon + \frac{1}{288} \left(\Gamma_\mu{}^{\nu\rho\lambda\sigma} - 8\delta_\mu^\nu \Gamma^{\rho\lambda\sigma} \right) F_{\nu\rho\lambda\sigma}\varepsilon\,. \tag{4.1}$$

and we will take the gamma-matrices to be

$$\begin{aligned} \Gamma_1 &= -i\,\Sigma_2 \otimes \gamma_9\,, \qquad \Gamma_2 = \Sigma_1 \otimes \gamma_9\,, \qquad \Gamma_3 = \Sigma_3 \otimes \gamma_9\,, \\ \Gamma_{j+3} &= \mathcal{I}_{2\times 2} \otimes \gamma_j\,, \quad j = 1, \ldots, 8\,, \end{aligned} \tag{4.2}$$

where the Σ_a are the Pauli spin matrices, $\mathbb{1}$ is the Identity matrix, and the γ_j are real, symmetric SO(8) gamma matrices. As a result, the Γ_j are all real, with Γ_1 skew-symmetric and Γ_j symmetric for $j > 2$. One also has:

$$\Gamma^{1\cdots\cdots 11} = \mathbb{1}\,,$$

where $\mathbb{1}$ will henceforth denote the 32×32 identity matrix.

Poincaré invariance parallel to the brane means that we can choose a frame in which the Killing spinors are independent of (t, x, y). It is instructive to start with the supersymmetry variations $\Gamma^1\,\delta\psi_1 = \Gamma^2\,\delta\psi_2 = \Gamma^3\,\delta\psi_3 = 0$. These equations are identical, and reduce to a single condition of the form

$$\left(x_1\Gamma^4 + y_1\Gamma^5 - y_2\Gamma^{1234} + x_2\Gamma^{1235} + x_3\Gamma^{4678} + y_3\Gamma^{5678} + x_4\Gamma^{491011} + y_4\Gamma^{59\,10\,11}\right)\varepsilon = 0\,. \tag{4.3}$$

One can easily check that there are 16 solutions, if and only if

$$\vec{x}\cdot\vec{x} = \vec{y}\cdot\vec{y}\,, \qquad \vec{x}\cdot\vec{y} = 0\,, \tag{4.4}$$

and otherwise there are no solutions. It is amusing to note that (4.4) defines the conifold. Remarkably, it turns out that the solutions to (4.3) all yield solutions to the complete set of Killing-spinor conditions. The matrix in (4.3) thus determines the entire family of solutions.

In practice, it is considerably simpler, and indeed more enlightening, to reduce the problem further by first defining

$$\mathcal{M}_\mu\,\varepsilon = \Gamma^\mu(\delta\psi_\mu - \partial_\mu\varepsilon)\,, \quad \text{with no sum on } \mu\,, \tag{4.5}$$

and then observing that the projection matrices

$$\begin{aligned} \Pi_1^\pm &= \frac{1}{2}\left(\mathbb{1}\pm\Gamma^{12}\right), & \Pi_2^\pm &= \frac{1}{2}\left(\mathbb{1}\pm\Gamma^{679\,10}\right), \\ \Pi_3^\pm &= \frac{1}{2}\left(\mathbb{1}\pm\Gamma^{689\,11}\right), & \Pi_4^\pm &= \frac{1}{2}\left(\mathbb{1}\pm\Gamma^{78\,10\,11}\right), \end{aligned} \tag{4.6}$$

all commute with the explicit forms of the operators $\mathcal{M}_\mu$. The projector Π_4 is redundant given Π_2 and Π_3, and the effect of these projectors is to reduce each of the gravitino variations to a set of eight 4×4 matrices. The Killing spinors then form two-dimensional eigenspaces of each of these eight matrices.

The solutions are most easily characterised in the following manner. Introduce the following functions:

$$\begin{aligned} h_0 &\equiv \frac{1}{2\cosh\alpha}\left(\sqrt{\frac{Y}{\widetilde{Y}}} + \sqrt{\frac{\widetilde{Y}}{Y}}\right), \\ h_1 &\equiv \sin\zeta\sinh\alpha\frac{\cos\theta}{\sqrt{Y}}\,, \qquad h_2 \equiv \sin\zeta\sinh\alpha\frac{\sin\theta}{\sqrt{\widetilde{Y}}}\,, \end{aligned} \tag{4.7}$$

and define the matrix

$$\mathcal{P} \equiv \frac{1}{2}\left(\mathbb{1} + h_0\,\Gamma^{123} + \Gamma^{45}\left(h_2\,\Gamma^{678} + h_1\,\Gamma^{9\,10\,11}\right)\right). \tag{4.8}$$

The functions, h_j, satisfy the identity

$$h_0^2 + h_1^2 + h_2^2 = 1\,, \tag{4.9}$$

and hence $\mathcal{P}$ is a projection matrix, satisfying $\mathcal{P}^2 = \mathcal{P}$. The solutions to $\delta\psi_\mu = 0$ for $\mu = 1,2,3$ are then given by the solutions to

$$\mathcal{P}\,\varepsilon_0 = 0\,. \tag{4.10}$$

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The gravitino variations parallel to the sphere are largely controlled by the symmetries. The only subtle issue is the SU(2) helicity projectors, and for this it is useful to recall that the situation for constant spinors on $\mathbb{R}^4$, but written in terms of polar coordinates. That is, consider the covariantly constant spinors

$$\nabla_\mu \varepsilon = 0\,, \tag{4.11}$$

for the metric

$$ds^2 = dr^2 + \frac{1}{4} r^2 \,\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right), \tag{4.12}$$

One finds that (4.11) may be written

$$dx^\mu \nabla_\mu \varepsilon = d\varepsilon - \frac{1}{8}\sigma_k \eta^k_{ab}\gamma^{ab}\varepsilon = 0\,, \tag{4.13}$$

where the η^k_{ab} are the self-dual 't Hooft matrices. The connection term in (4.13) thus acts on only one spinor helicity. Thus the solution to (4.13) are spinors that are of one helicity and are independent of the coordinates of S^3, and spinors of the other helicity that are the non-trivial solutions of (4.13), but whose dependence on angle can be obtained by using Lie transport of the spinor.

For the more complicated problem at hand we need to identify the correct helicity projector, and the rest is straightforward. To this end, define matrices $\mathcal{E}$ and $\widehat{\mathcal{E}}$ by

$$\begin{aligned}
\mathcal{E} &\equiv (h_1^2 + h_2^2)^{-1/2}\,(\cosh^2\alpha\,\sin^2 2\theta + \cos^2 2\theta)^{-1/2}\,\times \\
&\quad\times\,\left(h_2\,\Gamma^{678} + h_1\,\Gamma^{9\,10\,11}\right)\,\left(\cosh\alpha\,\sin 2\theta\,\Gamma^4 + \cos 2\theta\,\Gamma^5\right), \\
\widehat{\mathcal{E}} &\equiv \frac{\sin\theta}{\sqrt{Y}}\,\Gamma^{5678} - \frac{\cos\theta}{\sqrt{\widetilde{Y}}}\,\Gamma^{59\,10\,11} - \frac{\sin\zeta\,\sinh 2\alpha\,\sin\theta\,\cos\theta}{\sqrt{Y\,\widetilde{Y}}}\,\Gamma^5\,.
\end{aligned} \tag{4.14}$$

These matrices satisfy

$$\mathcal{E}^2 = \widehat{\mathcal{E}}^2 = \mathbb{1}\,, \qquad [\mathcal{E},\mathcal{P}] = 0\,, \qquad \left[\widehat{\mathcal{E}},\mathcal{P}\right] = 0\,, \qquad \mathcal{E}\,\varepsilon_0 = \widehat{\mathcal{E}}\,\varepsilon_0\,, \tag{4.15}$$

where ε_0 is a solution to (4.10). That is, both square to the identity, commute with the projector, $\mathcal{P}$ of (4.8), and they have equivalent actions of spinors that satisfy (4.10). We will only need one of the matrices, but we include the equivalent forms for completeness.

Introduce the helicity projectors, and helicity components of a spinor

$$\Pi_0^\pm \equiv \frac{1}{2}\,(\mathbb{1} \pm \mathcal{E})\,, \qquad \varepsilon^\pm \equiv \Pi_0^\pm\,\varepsilon \tag{4.16}$$

One can then check that the gravitino variations parallel to the spheres reduce to the condition

$$d\varepsilon + \frac{1}{2}\left(\left(\sigma_1\Gamma^{78} - \sigma_2\Gamma^{68} + \sigma_3\Gamma^{67}\right) + \left(\tilde\sigma_1\Gamma^{1011} - \tilde\sigma_2\Gamma^{911} + \tilde\sigma_3\Gamma^{910}\right)\right)\Pi_0^-\,\varepsilon = 0\,. \tag{4.17}$$

In arriving that this equation we have also made use of (4.10): we assume, henceforth that all our spinors indeed satisfy (4.10).

Since $\Pi_0^- \, \varepsilon^+ = 0$, it follows that such spinors solve the variation equations parallel to the spheres if and only if

$$\partial_\mu \, \varepsilon^+ = 0 \,, \qquad \mu = 6, \ldots, 11 \,. \tag{4.18}$$

That is, the ε^+ must be independent of the coordinates on the spheres.

The spinors, ε^-, of the opposite helicity also provide simple solutions. Define

$$\mathcal{R}_{AB}(\varphi) = \cos\left(\frac{1}{2}\varphi\right) \mathbb{1}_{AB} - \sin\left(\frac{1}{2}\varphi\right) \Gamma^{AB} \,, \tag{4.19}$$

and let

$$g \equiv \mathcal{R}_{67}(\varphi_3)\, \mathcal{R}_{68}(\varphi_1) \mathcal{R}_{67}(\varphi_2) \mathcal{R}_{9\,10}(\tilde{\varphi}_3) \mathcal{R}_{9\,11}(\tilde{\varphi}_1) \mathcal{R}_{9\,10}(\tilde{\varphi}_2) \,. \tag{4.20}$$

By construction one has

$$dg\, g^{-1} = -\frac{1}{2} \left(\left(\sigma_1 \, \Gamma^{78} - \sigma_2 \, \Gamma^{68} + \sigma_3 \Gamma^{67}\right) + \left(\tilde{\sigma}_1 \Gamma^{10\,11} - \tilde{\sigma}_2 \, \Gamma^{9\,11} + \tilde{\sigma}_3 \, \Gamma^{9\,10}\right)\right) , \tag{4.21}$$

and therefore the solution to the gravitino variation parallel to the spheres is generated by

$$\varepsilon^- = g \, \Pi_0^- \, \varepsilon_0 \,, \tag{4.22}$$

for some spinor, ε_0 that satisfies (4.10), and depends only upon r and θ.

The last step is to determine the r and θ dependence, and solve the corresponding gravitino variation equations. This turns out to be elementary: it is a theorem that if the ε_k are solutions to $\delta\psi_\mu = 0$ for $\delta\psi_\mu$ given by (4.1), then the bilinears

$$K^\mu_{ij} \equiv \bar{\varepsilon}_i \Gamma^\mu \varepsilon_j \,, \tag{4.23}$$

must be Killing vectors. This completely fixes the r and θ dependence of the spinors defined above. Indeed, it suffices to check that the first three components, K^μ_{ij}, $\mu = 1, 2, 3$ are all constants (translations parallel to the brane). Having thus normalised $\varepsilon^\pm$, one can then check that indeed all the gravitino variations vanish for the sixteen spinors given by solving (4.10), and using (4.16), (4.18) and (4.22).

It is, perhaps, useful to include one or two more explicit details. Recall that one can reduce the problem to 4×4 blocks using the projectors, (4.6). In a suitably chosen basis, these 4×4 blocks each yields two orthonormal solutions of the form (4.10):

$$\begin{aligned} \varepsilon_1 &= \frac{1}{\sqrt{2(h_0+1)}} \left(h_0 + 1, \quad 0, \quad h_1, \quad h_2 \right) , \\ \varepsilon_2 &= \frac{1}{\sqrt{2(h_0+1)}} \left(0, \quad h_0 + 1, \quad h_2, \quad -h_1 \right) . \end{aligned} \tag{4.24}$$

They are orthonormal in that $\varepsilon_1 \cdot \varepsilon_1 = \varepsilon_2 \cdot \varepsilon_2 = 1$, and $\varepsilon_1 \cdot \varepsilon_2 = 0$. (One uses (4.9) to verify this.) One also has $\varepsilon_2 = \Gamma^{45} \varepsilon_1$ for this pair of spinors.

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One can then apply $\Pi_0^{\pm}$ to these, and this generates $\varepsilon^{\pm}$ as independent linear combinations of ε_1 and ε_2. One must once again normalise these projected spinors. The result is algebraically messy, but it yields the Killing spinors. One can then get the other fourteen supersymmetries either by doing this in each of the 4×4 blocks, or simply by acting with $\Gamma^{23}, \Gamma^{67}, \Gamma^{78}$ and Γ^{68} on the $\varepsilon^{\pm}$ obtained in the first 4×4 block.

There is also a rather simple way to obtain the matrix $\mathcal{E}$: there is a linear combination of gravitino variations from which the tensor gauge field, F, cancels completely, leaving only connection terms. Specifically, one has

$$\mathcal{M}_1 + \mathcal{M}_6 + \mathcal{M}_9 = \frac{1}{4L}\Omega^2 \left(\left(h_2 \Gamma^{678} + h_1 \Gamma^{9\,10\,11} \right) + \left(\cosh\alpha \sin 2\theta \Gamma^4 + \cos 2\theta \Gamma^5 \right) \right) \varepsilon \,, \quad (4.25)$$

for the matrices defined in (4.5). This must vanish for the spinors that are independent of the coordinates on the spheres, and indeed the left-hand side of (4.25) is a multiple of $\Pi_0^- \equiv \frac{1}{2}(\mathbb{1} - \mathcal{E})$. From this one can obtain $\mathcal{E}$.

5. Dielectric rotations of the Coulomb branch flow

5.1 The Coulomb branch

For $\zeta = 0$, the flow defined in the previous sections lies in the purely scalar sector of supergravity, and must reduce to a standard "Coulomb branch flow," for which everything can be expressed in terms of a single harmonic function.

Setting $\zeta = 0$ simplifies things significantly: the transverse components of $A^{(3)}_{mnp}$ vanish, and one has $\widetilde{Y} = e^{-2\alpha} Y$. The pre-factor in front of the M2-brane section of the metric reduces to $H^{-2/3}$, where

$$H \equiv \frac{e^{\alpha} \sinh^3 \alpha}{Y|_{\zeta=0}} = \frac{\sinh^3 \alpha}{(e^{-\alpha} \sin^2 \theta + e^{\alpha} \cos^2 \theta)} \,. \quad (5.1)$$

This is the harmonic function, but the metric is in some peculiar coodinates adapted to gauged supergravity.

We therefore introduce new variables, according to

$$x = \frac{1}{\sqrt{(e^{2\alpha} - 1)}} \cos\theta \,, \qquad y = \frac{e^{\alpha}}{\sqrt{(e^{2\alpha} - 1)}} \sin\theta \,. \quad (5.2)$$

In terms of these, the metric becomes

$$ds_{11}^2 = H^{-2/3} k^2 (-dx_0^2 + dx_1^2 + dx_2^2) + 8 H^{1/3} L^2 \left(dx^2 + dy^2 + \frac{1}{4} x^2 \sum_{j=1}^{3} \sigma_j^2 + \frac{1}{4} y^2 \sum_{j=1}^{3} \tilde{\sigma}_j^2 \right) . \quad (5.3)$$

This is, of course, the requisite "harmonic" form, with flat metrics inside the parentheses.

In the new variables, the harmonic function takes the rather unedifying form

$$H = \frac{1}{16 x^2 y^2 (x^2 + y^2) \sqrt{\Delta}} \left((x^2 + y^2) \sqrt{\Delta} - \left((x^2 + y^2)^2 + (x^2 - y^2) \right) \right) . \quad (5.4)$$

where

$$\Delta \equiv (x^2 + (y-1)^2)\,(x^2 + (y-1)^2)\,. \tag{5.5}$$

However, by looking at the asymptotics of this for small x, one finds that

$$H \sim \frac{1}{4\,x^2}\,, \qquad \text{for } \ y^2 < 1\,.$$

This means that the M2-branes are spread out into a solid 4-ball, defined by $y^2 < 1$, with a constant density of branes throughout the ball. In particular, the function, $H(x,y)$ is given by

$$H(x,y) = \text{const.} \int_{z^2<1} \frac{d^4 z}{(x^2 + (\vec{y} - \vec{z})^2)^3}\,, \tag{5.6}$$

where $\vec{y}$ and $\vec{z}$ are vectors in $\mathbb{R}^4$.

The Killing spinor analysis simplifies in that one now has

$$\mathcal{P} \equiv \frac{1}{2}\left(\mathbb{1} + \Gamma^{123}\right). \tag{5.7}$$

If one writes $x = u\cos\psi$, $y = u\sin\psi$, then the matrix $\widehat{\mathcal{E}}$ becomes, in these new coordinates,

$$\widehat{\mathcal{E}} \equiv \sin\psi \Gamma^{5678} - \cos\psi \Gamma^{59\,10\,11}\,. \tag{5.8}$$

It is not completely trivial because the flat metric on $\mathbb{R}^8$ metric has been written in terms of angular coordinates. The earlier comments about normalising the Killing spinors amount to putting the proper power of the harmonic function into the spinor.

5.2 Dielectric rotations

The general solution presented in sections 2 and 3 has a very non-trivial form, and yet it is related via a simple $U(1) \subset SU(8)$ duality rotation, (2.3), to the much simpler, and well-understood "harmonic" M2-brane distribution described above. Thus, in terms of the eleven-dimensional background, the SU(8) duality rotation, (2.3), has a highly non-trivial effect that we will now attempt to characterise.

Under the U(1) rotation, the harmonic function (5.1) appears to split into two components, which we will denote by H_1 and H_2, with $H_j \sim Y_j^{-1}$. These component functions do not seem to be harmonic within any obvious flat metric, and so we have not defined them precisely, but one would like to choose them so that

$$\frac{\Omega^2}{\sinh^2(\alpha)} = (H_1\, H_2)^{-1/3}\,. \tag{5.9}$$

Note that the left-hand side of (5.9) is the coefficient of the M2-brane sections of the metric defined by (3.10). Also note that this is consistent with (5.1) for $\zeta = 0$. These "pseudo-harmonic" functions also appear in the internal parts of the metric parallel to the spheres. Indeed, the eleven-metric has the schematic form

$$\begin{aligned} ds_{11}^2 \sim\ & (H_1\, H_2)^{-1/3}\,(-dx_0^2 + dx_1^2 + dx_2^2) + h(r,\theta)(dr^2 + 4\,L^2\,d\theta^2) + \\ & + H_1^{2/3} H_2^{-1/3} \left(\sum_{j=1}^{3} \sigma_j^2\right) + H_1^{-1/3} H_2^{2/3} \left(\sum_{j=1}^{3} \tilde{\sigma}_j^2\right). \end{aligned} \tag{5.10}$$

The way that the pseudo-harmonic functions appear is suggestive of M5-branes wrapping the 3-spheres, and intersecting over what was the original M2-brane. This suggestion is strongly supported by the form of $A^{(3)}_{mnp}$: there is still an "electric" M2-brane component parallel in the 123 directions, while the components parallel to the 678 and 9 10 11 directions are consistent with M5-branes in the 1239 10 11 and 123678 directions respectively. Note also that the pseudo-harmonic functions, and *not* their inverses, appear as the components of $A^{(3)}_{mnp}$ parallel to the spheres. We thus have a distribution of M5-branes wrapping the spheres in the 679 and 9 10 11 directions. The relative sign for the corresponding components of $A^{(3)}_{mnp}$ in (3.13) make it tempting to think of these branes as being M5-branes and anti-M5-branes, but such a distinction is largely a matter of orientation.

This picture is particularly evident in the form of the Killing spinors. The crucial step in finding the Killing spinors was to identify the projection matrix, $\mathcal{P}$, that defines the relevant sixteen-dimensional subspace in which they live. It turns out that the general projection, $\mathcal{P}$, can be obtained via an SU(8) rotation of the simple projector, (5.7), for the "harmonic" solution.

Define

$$\mathcal{R} = \frac{1}{\sqrt{2(h_0+1)}} \big((h_0+1)\mathbb{1} - h_1 \Gamma^{678} + h_2 \Gamma^{9\,10\,11} \big) . \tag{5.11}$$

One can then check that the projection matrix, $\mathcal{P}$, of (4.8) is related to the projection, $\frac{1}{2}(\mathbb{1}+\Gamma^{123})$, of (5.7) via

$$\mathcal{P} = \frac{1}{2} \mathcal{R} \left(\mathbb{1} + \Gamma^{123} \right) \mathcal{R}^{-1} . \tag{5.12}$$

The matrix, $\mathcal{R}$, is a real rotation matrix in SO(32). Indeed, since it only acts on the eight-dimensional "internal space," we may think of it as an SO(16) matrix acting on the 8_s and 8_c spinors of the internal space. Since $\mathcal{R}$ involves the triple products of gamma matrices, and these flip SO(8) helicity, $\mathcal{R}$ actually lives within the SU(8) subgroup of SO(16). Since $\mathcal{R}$ is an SU(8) matrix that preserves SO(4)$\times$SO(4) symmetry, and rotates the Killing spinor projector from the harmonic flow to the general flow, it is natural to think of $\mathcal{R}$ as the eleven-dimensional analogue of the SU(8) rotation (2.3). This is precisely the local SU(8) action that plays a crucial role in the consistent truncation proof of [10].

The other non-trivial matrix needed to obtain the Killing spinors is the helicity projector, $\mathcal{E}$. The Killing spinors lie in either singlets or doublets of the $(\mathrm{SU}(2))^4$ symmetry of the solution, and the role of $\mathcal{E}$ is to sort the solutions to (4.10) into linear combinations that are either singlets or doublets of the SU(2)'s. As remarked earlier, the form of $\mathcal{E}$ can be found by taking the linear combination of gravitino variations from which the tensor gauge field cancels. However, it is also highly constrained, but not quite uniquely determined by the fact that it must commute with the rotation, $\mathcal{R}$, and with the projector, $\mathcal{P}$. It may also be shifted by multiples of $\mathcal{P}$.

In terms of brane distributions, the effect of $\mathcal{R}$ is quite dramatic. The unrotated projector, (5.7), is the standard Dirichlet projector imposed by the presence of the branes. Additional branes usually reduce supersymmetry by imposing more projection conditions, however the dielectric distribution discovered here simply deforms the standard projector in the directions of the fluxes generated by the dielectric M5-branes.

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6. Final comments

While there are many solutions that can be generated using gauged supergravity theories, we find the solution presented here especially interesting. It has the maximal amount of supersymmetry possible for a brane solution, and yet consists of a complicated combination of M2-branes and M5-branes. As was also seen in [8], the solution presented here shows how a relatively simple duality symmetry of gauged supergravity can translate into a very non-trivial symmetry of the complete eleven-dimensional theory, while preserving all the supersymmetries. More generally, it is still an open problem to classify supersymmetric compactifications in the presence of fluxes, and from this paper it is evident that even maximally supersymmetric flows have a rather richer structure than one might have otherwise expected.

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